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Exact large amplitude capillary waves on sheets of fluid

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We generalize Crapper's exact solution for capillary waves on fluid of infinite depth. We find two finite-depth solutions involving elliptic functions. We show they can also be interpreted as large amplitude symmetrical and antisymmetrical waves on a fluid sheet. Particularly interesting are the waves obtained from our solution in the limit when the fluid sheet is extremely thin.

1. Introduction

Small amplitude capillary waves on fluid surfaces form a well-known part of the classical theory of hydrodynamics, and have been studied in great detail in the past. For example, such waves on the plane surface of fluids of both finite and infinite depth and on the cylindrical surface of a uniform fluid jet have been treated. Also, Taylor (1959) has analysed the small amplitude capillary waves which occur on a thin fluid sheet. He has observed that both symmetrical and antisymmetrical waves of this type can exist, and he was able to produce both of them experimentally, using fluid sheets as thin as 5–100 μm.

Large amplitude capillary waves are not as well known. A remarkable exact wave solution in this category was presented by Crapper (1957). It represented a particular class of large amplitude capillary waves on fluid of infinite depth. Further, Crapper's solution had the virtue of extreme simplicity, involving only elementary functions. At that time, Crapper made the following statement: "there is also an exact solution if the fluid has finite uniform depth" but "the analysis needed is too elaborate to make this solution worthwhile". In view of its importance, it is surprising that this statement has gone untested for almost twenty years.

In the present paper we shall examine this point, and show that Crapper's remark is inaccurate in two respects. First, there is not *one* generalization but *two*. They turn out to be large amplitude versions of Taylor's symmetrical and antisymmetrical sheet waves. Secondly, the analysis is by no means complicated. Despite the presence of elliptic functions in the solutions, the analysis needed is no more than a straightforward generalization of Crapper's own approach. In particular, the dispersion formulae which give the wave velocity turn out to be quite simple.

Just as in Crapper's case, we shall find that there is a maximum amplitude allowed for the waves. For very large amplitudes the solutions become multivalued, and hence are no longer suitable on physical grounds. Some new and particularly interesting results will be obtained in another limiting case, when the sheet is allowed to become very thin. The antisymmetrical waves then reduce to a particular solution of the simpler (but still nonlinear) problem of large amplitude waves on a uniform elastic string. The symmetrical waves become what may be regarded as a new type of wave: an *essentially* nonlinear wave, in which the phase velocity is proportional to the square root of the amplitude.

2. Basic equations

Consider a capillary wave progressing to the right with phase velocity c. Let (x, y) be a system of Cartesian co-ordinates moving with the waves, so that the flow appears to be steady. Let x be measured to the left (upstream) and let y be measured vertically downwards from the undisturbed surface. To obtain an exact solution we must ignore both the viscosity of the fluid and the induced motion of the surrounding atmosphere (even though there is no reason to expect that these effects will be unimportant in practice; cf. Crapper, Dombrowski & Pyott 1975).

We must solve Laplace's equation

$$\nabla^2 \hat{\Phi} = 0 \tag{1}$$

for the velocity potential, $\hat{\Phi}$, subject to the boundary conditions

$$\tfrac{1}{2}\rho(\hat{\Phi}_x^2 + \hat{\Phi}_y^2) + p - \rho g y = \tfrac{1}{2}\rho c^2, \tag{2}$$

$$\hat{\Phi}_y + \eta_x \hat{\Phi}_x = 0 \tag{3}$$

at the free surface $y = \eta(x)$.

We make use of the standard formulation in terms of complex variables:

$$\left.\begin{aligned} z = x + iy, \quad w = \hat{\Phi} + i\hat{\Psi}, \\ dw/dz = u - iv = \hat{q}\, e^{-i\theta}, \end{aligned}\right\} \tag{4}$$

where $\hat{\Psi}$ is the stream function, u and v the velocity components, and $\hat{q}$ and θ the speed and direction of flow. We now take $(\hat{\Phi}, \hat{\Psi})$ as the independent co-ordinates in place of (x, y), since this transforms the free boundary into a fixed line $\hat{\Psi} = 0$.

Defining τ by the relation

$$\hat{q} = e^{\tau}, \tag{5}$$

we see that $\tau - i\theta$ must be an analytic function of $\hat{\Phi} + i\hat{\Psi}$, implying the Cauchy–Riemann relations

$$\theta_{\hat{\Phi}} = \tau_{\hat{\Psi}}, \quad \theta_{\hat{\Psi}} = -\tau_{\hat{\Phi}}. \tag{6}$$

If p_0 is the atmospheric pressure (assumed constant) and the fluid has a prescribed surface tension T, then the effective pressure at the surface of the liquid is

$$p = p_0 + \kappa T, \tag{7}$$

where κ is the curvature of the surface streamline:

$$\kappa = \frac{d\theta}{ds} = \frac{d\hat{\Phi}}{ds}\theta_{\hat{\Phi}} = e^{\tau}\tau_{\hat{\Psi}} = \hat{q}_{\hat{\Psi}}. \tag{8}$$

Setting $p_0 = 0$ for convenience, the remaining boundary condition, equation (2), becomes

$$\tfrac{1}{2}\rho\hat{q}^2 + T\hat{q}_{\hat{\Psi}} = \tfrac{1}{2}\rho c^2.$$

Alternatively, in terms of the dimensionless variables

$$q = c^{-1}\hat{q}, \quad \Psi = (\rho c/T)\hat{\Psi}, \quad \Phi = (\rho c/T)\hat{\Phi} \tag{9}$$

it is

$$q_{\Psi} = \tfrac{1}{2}(1-q^2) \quad \text{at} \quad \Psi = 0. \tag{10}$$

Following Crapper, we arbitrarily assume a more general condition to hold *throughout* the fluid:

$$q_{\Psi} = \tfrac{1}{2}(1-q^2)f(\Psi), \quad f(0) = 1. \qquad (11), (12)$$

Physically, the import of this assumption is that *any* streamline in the moving fluid has potentially all the properties required of a free boundary. If the fluid on one side of a streamline were suddenly removed, the fluid on the other side could continue its flow undisturbed. (However the normalization condition (12) would not in general be satisfied on the new surface, and there would be required a different surface tension, constant but not equal to T.) Clearly the assumption is a restrictive one and many other possible types of motion will be excluded by it.

Defining a function $P(\Psi)$ by

$$P(\Psi) = \exp\left(\int f(\Psi)\, d\Psi\right), \tag{13}$$

the normalization condition (12) becomes

$$P'(0)/P(0) = 1. \tag{14}$$

Equation (11) may now be integrated with respect to Ψ, yielding

$$\frac{1+q}{1-q} = \frac{P(\Psi)}{Q(\Phi)} \equiv \xi, \tag{15}$$

where $Q(\Phi)$ is the integration constant.† Thus

$$q = \frac{P(\Psi)-Q(\Phi)}{P(\Psi)+Q(\Phi)}. \tag{16}$$

All that remains is to satisfy Laplace's equation. In terms of ξ it is a nonlinear equation:

$$(\xi^2-1)\nabla^2\xi = 2\xi\nabla\xi \,.\, \nabla\xi. \tag{17}$$

Inserting (15) into (17) we get

$$(P^2-Q^2)[P''/P-Q''/Q] = 2[P'^2+Q'^2]. \tag{18}$$

In each term the primes denote differentiation with respect to the corresponding argument. Separation of variables is accomplished by differentiating twice more, once with respect to Ψ and once with respect to Φ. We find eventually in this manner that P and Q must be solutions of the following ordinary nonlinear differential equations:

$$P'^2 = c_1 + c_2P^2 + c_3P^4, \tag{19}$$

$$Q'^2 = -c_1 - c_2Q^2 - c_3Q^4, \tag{20}$$

where c_1, c_2 and c_3 are all arbitrary separation constants. Since we want our solu-

† Our functions $P(\Psi)$ and $Q(\Phi)$ correspond to Crapper's functions $X(\Psi)$ and $Y(\Phi)$.

tion to represent a wave of finite amplitude, the function Q must be bounded and periodic. This requirement does place some restrictions on c_1, c_2 and c_3, as we shall see.

3. Solutions

In general, the solutions of (19) and (20) involve elliptic functions. (An excellent reference for these is Byrd & Friedman 1971). In fact all twelve of the Jacobi elliptic functions satisfy differential equations of this type. The numerous possibilities can be resolved by examination of the quadratic form

$$g(x) = c_1 + c_2 x + c_3 x^2.$$

To obtain a solution which is real we must restrict consideration to those ranges of $x > 0$ such that $g(x) > 0$ for (19), or $g(x) < 0$ for (20). Since Q must be bounded, we require also that the range of x for (20) be bounded. These conditions reduce the number of possible cases to three, which are illustrated in figure 1. (Case III may of course be obtained as a limit of case I or case II, but it is just as simple to treat it separately.)

For case I the general solutions of (19) and (20) may be written as

$$P = \mathrm{nd}\,(\psi, k'), \quad Q = \mathrm{cd}\,(\phi, k),$$

where k is the modulus of the elliptic function, k' is the complementary modulus and

$$\psi = A\Psi + B, \quad \phi = A\Phi. \tag{21), (22}$$

The three original arbitrary constants c_1, c_2 and c_3 have now been replaced by the equivalent set of constants A, B and k. (Simple algebraic equations may be given to relate the two sets but they are never actually needed.)

To make the notation more concise we shall not write out each time the dependence of these functions on the modulus. In all that follows, it is to be understood that functions of ϕ have modulus k and functions of ψ have modulus k'.

An alternative form exists for the case I solution:

$$P = \mathrm{dn}\,\psi, \quad Q = k\,\mathrm{cd}\,\phi.$$

This form may be obtained from the first via the identity

$$\mathrm{dn}\,\psi \equiv k\,\mathrm{nd}\,(K' - \psi)$$

(where $K'(k')$ is the complete elliptic integral) along with a suitable redefinition of A and B. In fact all that we have done is to reverse the labelling of the streamlines, so that ψ increases upwards rather than downwards. Although the two versions are totally equivalent, it is still very useful to consider them both. We refer to them as cases Ia and Ib.

All possibilities for P and Q have been listed in table 1. From (5) and (15), we find

$$\tau = \ln\left(c\frac{P-Q}{P+Q}\right). \tag{23}$$

The function θ must next be determined by solving simultaneously the Cauchy–

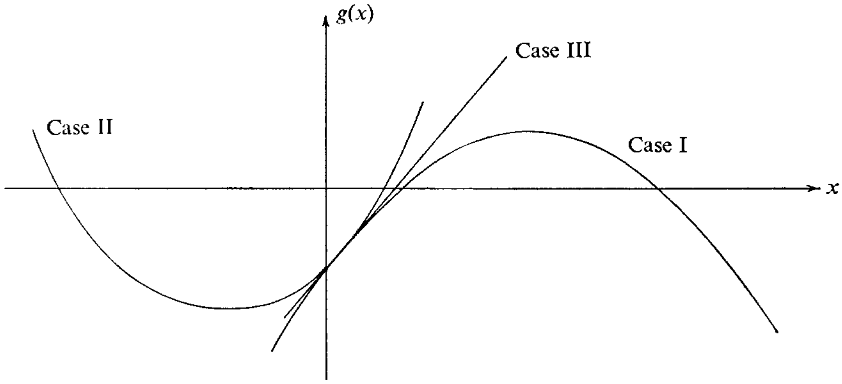

FIGURE 1. The function $g(x)$.

Riemann relations (6). We find that in each case θ may be taken to have the form

$$\theta = 2\tan^{-1}(S(\phi)/R(\psi)). \tag{24}$$

Using this ansatz, the Cauchy–Riemann equations reduce to

$$\frac{P^2-Q^2}{R^2+S^2} = \frac{P'Q}{RS'} = -\frac{PQ'}{R'S}, \tag{25}$$

implying

$$-P'R'/PR = Q'S'/QS. \tag{26}$$

Since one side is a function of ψ while the other is a function of ϕ, both are equal to some constant, α say. The solutions for R, S and α are listed in table 1.

Now we can write out the velocity components:

$$u = q\cos\theta = c\left(\frac{P-Q}{P+Q}\right)\left(\frac{R^2-S^2}{R^2+S^2}\right) = \frac{c}{\alpha}\frac{P'Q'}{(P+Q)^2}\left(\frac{R^2-S^2}{RS}\right), \tag{27}$$

$$v = q\sin\theta = c\left(\frac{P-Q}{P+Q}\right)\left(\frac{2RS}{R^2+S^2}\right) = \frac{2c}{\alpha}\frac{P'Q'}{(P+Q)^2}, \tag{28}$$

and finally the displacements themselves, using (4):

$$\begin{aligned} x &= -\frac{T}{\rho cA}\int\frac{v\,d\psi}{q^2} = \frac{T}{\rho cA}\int\frac{u\,d\phi}{q^2} \\ &= \frac{T}{\rho c^2\alpha A}\left[\frac{2Q'}{P-Q}+\int\frac{Q''}{Q}\,d\phi\right], \end{aligned} \tag{29}$$

$$\begin{aligned} y &= \frac{T}{\rho cA}\int\frac{u\,d\psi}{q^3} = \frac{T}{\rho cA}\int\frac{v\,d\phi}{q^2} \\ &= \frac{T}{\rho c^2\alpha A}\left[\frac{2P'}{P-Q}-\int\frac{P''}{P}\,d\psi\right]. \end{aligned} \tag{30}$$

The integrals remaining in these last two equations can be expressed in terms of the complete elliptic integral $E(u)$. They have been listed for each case in table 2.

Before proceeding to the analysis of the solutions, we should point out some

Case	P	Q	R	S	α
Ia	$\mathrm{nd}\,\psi$	$\mathrm{cd}\,\phi$	$\mathrm{sc}\,\psi$	$\mathrm{sn}\,\phi$	$-k'^2$
Ib	$\mathrm{dn}\,\psi$	$k\,\mathrm{cd}\,\phi$	$\mathrm{cs}\,\psi$	$k\,\mathrm{sn}\,\phi$	$-k'^2$
IIa	$\mathrm{nc}\,\psi$	$\mathrm{cn}\,\phi$	$\mathrm{sd}\,\psi$	$\mathrm{sd}\,\phi$	-1
IIb	$\mathrm{ds}\,\psi$	$k\,\mathrm{cn}\,\phi$	$\mathrm{cn}\,\psi$	$k\,\mathrm{sd}\,\phi$	-1
III	$\cosh\psi$	$\cos\phi$	$\sinh\psi$	$\sin\phi$	-1

TABLE 1

Case	$\int P''/P\,d\psi$	$\int Q''/Q\,d\phi$
Ia	$(1+k^2)\psi-2E(\psi)+2k'^2\,\mathrm{sn}\,\psi\,\mathrm{cd}\,\psi$	$k'\phi-2E(\phi)+2k^2\,\mathrm{sn}\,\phi\,\mathrm{cd}\,\phi$
Ib	$(1+k^2)\psi-2E(\psi)$	
IIa	$\psi-2E(\psi)+2\,\mathrm{dn}\,\psi\,\mathrm{sc}\,\psi$	$\phi-2E(\phi)$
IIb	$\psi-2E(\psi)-2\,\mathrm{dn}\,\psi\,\mathrm{cs}\,\psi$	
III	ψ —	$-\phi$

TABLE 2

close relationships that exist among them. An obvious one is that case III may indeed be obtained from cases Ia and IIa in the limit $k\to 0$.

Another relationship, which is not so obvious, connects cases I and II directly. Case II may be formally obtained from case I via the transformation (Byrd & Friedman 1971, p. 38)

$$\left.\begin{gathered}\psi\to k'\psi,\quad \mathrm{sn}\,\psi\to k'\,\mathrm{sn}\,\psi,\quad \mathrm{cn}\,\psi\to\mathrm{dn}\,\psi,\quad \mathrm{dn}\,\psi\to\mathrm{cn}\,\psi,\\ E(\psi)\to[E(\psi)-k^2\psi]/k',\end{gathered}\right\}\tag{31}$$

$$\left.\begin{gathered}\phi\to k'\phi,\quad \mathrm{sn}\,\phi\to k'\,\mathrm{sn}\,\phi,\quad \mathrm{cn}\,\phi\to\mathrm{cd}\,\phi,\quad \mathrm{dn}\,\phi\to\mathrm{nd}\,\phi,\\ E(\phi)\to[E(\phi)-k^2\,\mathrm{sn}\,\phi\,\mathrm{cd}\,\phi]/k'.\end{gathered}\right\}\tag{32}$$

Needless to say, this is only a mathematical transformation, and does not prevent the solutions from being physically distinct.

4. Interpretation

Case I

We should now like to discuss the physical properties of these new waves. Using case Ib as an example, we write out the displacements explicitly:

$$x=\frac{T}{\rho c^2Ak'^2}\left[2E(\phi)-k'^2\phi-2k^2\,\mathrm{sn}\,\phi\,\mathrm{cd}\,\phi+\frac{2kk'^2\,\mathrm{sd}\,\phi\,\mathrm{nd}\,\phi}{\mathrm{dn}\,\psi-k\,\mathrm{cd}\,\phi}\right],\tag{33}$$

$$y=\frac{T}{\rho c^2Ak'^2}\left[(1+k^2)\psi-2E(\psi)+\frac{2k'^2\,\mathrm{sn}\,\psi\,\mathrm{cn}\,\psi}{\mathrm{dn}\,\psi-k\,\mathrm{cd}\,\phi}\right].\tag{34}$$

Recall that sn and cn range between ± 1 with period $4K$, while dn ranges between 1 and k' with period $2K$. The increase in x over one period of ϕ determines the wavelength:

$$\lambda = x(\phi = 4K) - x(\phi = 0) = \frac{4T}{\rho c^2 A k'^2}[2E - k'^2 K], \tag{35}$$

where $E(k)$ is the complete elliptic integral of the second kind.

The constant A is determined by the normalization condition (14) at the surface streamline (which now carries the label $\psi = B$). For case I b this condition is

$$1 = P'(B)/P(B) = -Ak'^2 \operatorname{sn} B \operatorname{cd} B. \tag{36}$$

Assuming that B has been chosen to lie in the first quadrant, A must be negative. Whereas the original Ψ and Φ increase downwards and to the right, a negative value of A implies that ψ and ϕ increase upwards and to the left. The same conclusion holds for case II b. Cases I a, II a and III all turn out to have an A which is positive. This is exactly as expected, since the labelling of streamlines was reversed when the (b) cases were being derived.

Returning to case I b, let us examine the expression (34) for y. We see that $\phi = 0$ corresponds to a crest and $\phi = 2K$ to a trough. If we follow an arbitrary streamline, the total variation in y will be

$$\Delta y = \frac{2T}{\rho c^2 A}\left[\frac{\operatorname{sn}\psi \operatorname{cn}\psi}{\operatorname{dn}\psi - k} - \frac{\operatorname{sn}\psi \operatorname{cn}\psi}{\operatorname{dn}\psi + k}\right] = \frac{4Tk}{\rho c^2 A k'^2} \operatorname{sc}\psi. \tag{37}$$

On the surface streamline, Δy gives us the trough-to-crest wave amplitude

$$a = (4Tk/\rho c^2 A k'^2) \operatorname{sc} B. \tag{38}$$

Below the surface Δy decreases steadily from this maximum value. It passes through zero at the 'centre streamline' $\psi = 0$ and is negative below it. Since Δy would become infinite at $\psi = -K'$, *the fluid must have a lower boundary* somewhere before that point is reached.

One possibility is that the case I waves occur on a fluid of finite depth, with a fixed bottom surface at $\psi = 0$. Note that we have $R = \infty$ on this surface, implying $\theta = 0$. However, since $P = 1$, the speed q is not constant. If the waves were viewed as moving by with the phase velocity c, the fluid near the bottom would not be at rest, but would slip back and forth to some degree. Of course this is not unexpected, since the usual gravity wave solutions for finite depth have the same property.

The other possibility to consider is that the fluid has a second free boundary. Assuming that the surface tension has the same value T on both surfaces, the lower boundary must occur on the streamline which is symmetrically located relative to the centre-line, namely $\psi = -B$. Under this interpretation, case I waves can be regarded as a nonlinear version of Taylor's symmetrical waves on a fluid sheet. The depth or thickness h, measured from the centre-line to a trough on the surface, may be obtained from (34) as

$$h = (T/\rho c^2 A k'^2)\,[(1+k^2)\,B - 2E(B) + 2 \operatorname{sc} B(\operatorname{dn} B - k)]. \tag{39}$$

Equations (35), (36), (38) and (39) give the phase velocity, wavelength, amplitude and depth parametrically in terms of A, B and k. The dispersion formula may be derived by eliminating A, B and k among these equations. Letting $\kappa = 2E - k'^2K$, we find

$$c = \left(\frac{4T\kappa}{\rho\lambda}\right)^{\frac{1}{2}} \left(1+\frac{\kappa^2 a^2}{\lambda^2}\right)^{-\frac{1}{4}} \left(1+\frac{k^2\lambda^2}{\kappa^2 a^2}\right)^{-\frac{1}{4}}. \tag{40}$$

For each value of k we thus have a simple algebraic relation between c, λ and a. Unfortunately it is not possible to eliminate k explicitly between (39) and (40). We must retain k, and regard it as a parameter which expresses the influence of the fluid's finite depth.

Cases II *and* III

A similar analysis of case IIb leads to the following equations:

$$x = \frac{T}{\rho c^2 A}\left[2E(\phi) - \phi + \frac{2k \operatorname{sn}\phi \operatorname{dn}\phi}{\operatorname{ds}\psi - k\operatorname{cn}\phi}\right], \tag{41}$$

$$y = \frac{T}{\rho c^2 A}\left[-2E(\psi) + \psi - 2\operatorname{dn}\psi \operatorname{cs}\psi + \frac{2\operatorname{ns}\psi \operatorname{cs}\psi}{\operatorname{ds}\psi - k\operatorname{cn}\phi}\right]. \tag{42}$$

$$\lambda = (4T/\rho c^2 A)[2E - K], \tag{43}$$

$$1 = -A \operatorname{cs} B \operatorname{nd} B, \tag{44}$$

$$\Delta y = \frac{4Tk}{\rho c^2 A}\operatorname{nc}\psi, \quad a = \frac{4Tk}{\rho c^2 A}\operatorname{nc} B, \tag{45), (46}$$

$$h = (T/\rho c^2 A)[B - 2E(B) + 2\operatorname{sn} B \operatorname{dc} B - 2k(\operatorname{nc} B - 1)], \tag{47}$$

$$\kappa = 2E - K, \tag{48}$$

$$c = \left(\frac{4T\kappa}{\rho\lambda}\right)^{\frac{1}{2}} \left(k'^2+\frac{\kappa^2 a^2}{\lambda^2}\right)^{-\frac{1}{4}} \left(1-\frac{k^2\lambda^2}{\kappa^2 a^2}\right)^{-\frac{1}{4}}. \tag{49}$$

In this instance we see from (45) that Δy reaches a non-zero minimum at $\psi = 0$, and is symmetric about that point. Therefore the most suitable interpretation here is the one with a second free boundary at $\psi = -B$. This solution is thus the large amplitude version of Taylor's antisymmetrical sheet waves.

Case III is the one previously treated by Crapper, but the results are also listed here for comparison:

$$x = \frac{T}{\rho c^2 A}\left[\phi + \frac{2\sin\phi}{\cosh\psi - \cos\phi}\right], \tag{50}$$

$$y = \frac{T}{\rho c^2 A}\left[\psi - \frac{2\sinh\psi}{\cosh\psi - \cos\phi}\right], \tag{51}$$

$$\lambda = 2\pi T/\rho c^2 A, \quad 1 = A\tanh B, \tag{52), (53}$$

$$\Delta y = \frac{4T}{\rho c^2 A}\operatorname{cosech}\psi, \quad a = \frac{4T}{\rho c^2 A}\operatorname{cosech} B, \tag{54),(55}$$

$$c = \left(\frac{2\pi T}{\rho\lambda}\right)^{\frac{1}{2}} \left(1+\frac{\pi^2 a^2}{\lambda^2}\right)^{-\frac{1}{4}}. \tag{56}$$

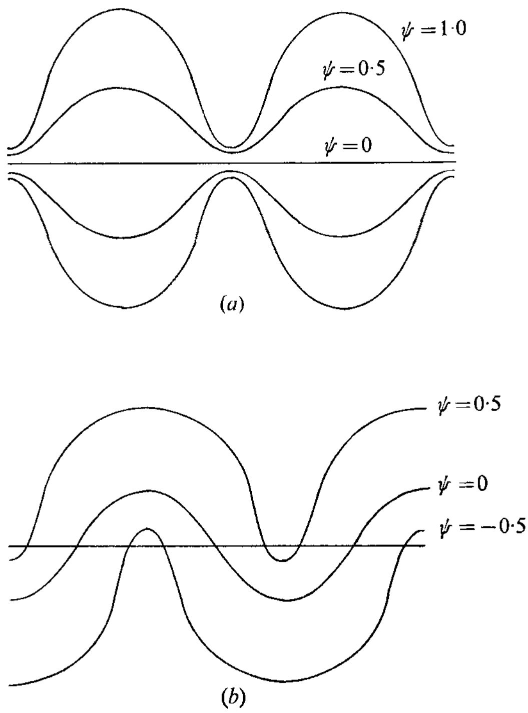

FIGURE 2. Typical wave profiles. $k = 0{\cdot}5$. (a) Case I. (b) Case II.

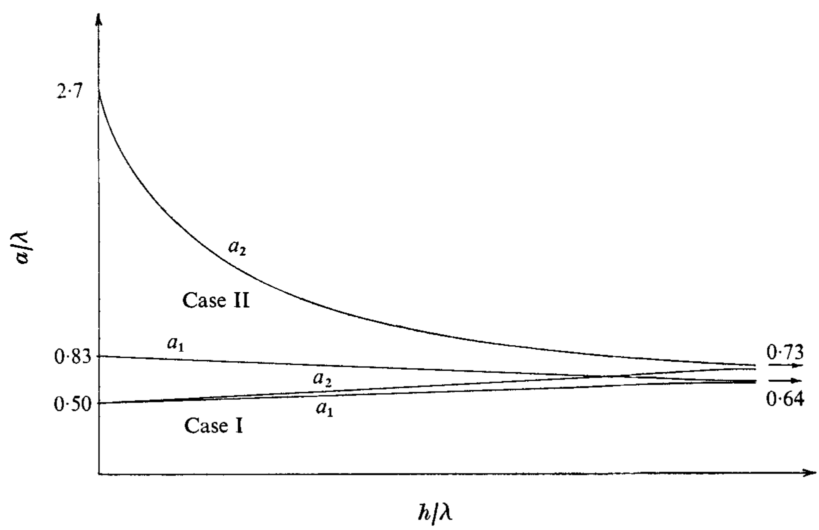

FIGURE 3. Critical amplitudes.

Wave profiles

Typical wave profiles have been obtained by computer and are shown in figure 2. In all three cases, when the amplitude reaches a critical value a_1, the surface becomes vertical at certain points. When it reaches or exceeds a second critical value a_2, the surface intersects itself and the solution can no longer be given a reasonable physical interpretation. The values of $a_1(k)$ and $a_2(k)$ have been obtained, and are shown in figure 3.

5. Limiting cases

The limit $k \to 0$

To support the interpretations we have just given, we can easily examine the small amplitude limit of these exact solutions. In particular we can show that the dispersion formulae correctly reduce to the well-known results of the small amplitude approximation.

From (40) and (49), if $a/\lambda \to 0$ while h/λ stays finite, we must necessarily have $k \to 0$. Recalling that all functions of ψ and B refer to the complementary modulus k', we have in this limit

$$\left.\begin{array}{lll} \operatorname{sn} B \to \tanh B, & \operatorname{cn} B \to \operatorname{sech} B, & \operatorname{dn} B \to \operatorname{sech} B, \\ E(B) \to \tanh B, & 2E - K \to \tfrac{1}{2}\pi, & 2E - k'^2 K \to \tfrac{1}{2}\pi. \end{array}\right\} \tag{57}$$

Therefore we have for case I, from (35) and (38)–(40),

$$\frac{a}{\lambda} \approx \frac{2k}{\pi} \sinh B, \quad \frac{h}{\lambda} \approx \frac{B}{2\pi}, \quad c^2 \approx \left(\frac{2\pi T}{\rho\lambda}\right) \tanh B, \tag{58}$$

and upon eliminating B we retrieve the familiar dispersion formula

$$c^2 = \left(\frac{2\pi T}{\rho\lambda}\right) \tanh\left(\frac{2\pi h}{\lambda}\right). \tag{59}$$

Similarly, for case II, from (43), (46), (47) and (49),

$$\frac{a}{\lambda} \approx \frac{2k}{\pi} \cosh B, \quad \frac{h}{\lambda} \approx \frac{B}{2\pi}, \quad c^2 \approx \left(\frac{2\pi T}{\rho\lambda}\right) \coth B, \tag{60}$$

which gives

$$c^2 \approx \left(\frac{2\pi T}{\rho\lambda}\right) \coth\left(\frac{2\pi h}{\lambda}\right). \tag{61}$$

The limit $B \to 0$

With the possession of these exact finite-depth solutions, we are also in a position to examine a more interesting limiting case: that of shallow-water theory, in which both $a/\lambda \to 0$ and $h/\lambda \to 0$ but a/h stays finite. This limit represents capillary waves on a fluid sheet which is very thin. It is also equivalent to letting $B \to 0$ with k fixed.

Consider case I. From the same equations, (35) and (38)–(40), with $B \to 0$,

$$\frac{a}{\lambda} \approx \frac{kB}{\kappa}, \quad \frac{h}{a} \approx \frac{(1-k)^2}{4k}, \quad c^2 \approx \frac{4T\kappa}{\rho\lambda} B. \tag{62}$$

Eliminating B and k we find

$$c^2 = (4T\kappa^2/\rho\lambda^2)\{a+2h+2[h(h+a)]^{\frac{1}{2}}\}. \tag{63}$$

The shape of the surface in this limit is given by

$$\left.\begin{aligned} x &= (\lambda/4\pi)\,[2E(\phi)-k'^2\phi+2k\,\mathrm{sn}\,\phi],\\ y &= (a/4k)\,[\mathrm{dn}\,\phi+k\,\mathrm{cn}\,\phi]^2. \end{aligned}\right\} \tag{64}$$

This is a well-behaved non-intersecting curve for all values of k.

Next consider case II. In the thin-sheet limit

$$q = c\left(\frac{\mathrm{ds}\,B-k\,\mathrm{cn}\,\phi}{\mathrm{ds}\,B+k\,\mathrm{cn}\,\phi}\right)$$

and as $B\to 0$ the speed becomes $q = c =$ constant. This further implies that the distance between neighbouring streamlines is constant, and hence the entire sheet maintains constant thickness. From (43), (46), (47) and (49),

$$\frac{a}{\lambda}\approx\frac{k}{\kappa},\quad \frac{h}{\lambda}\approx\frac{B}{4\kappa},\quad c^2\approx\frac{4T\kappa}{\rho\lambda B}, \tag{65}$$

giving the dispersion formula

$$c^2 = T/\rho h. \tag{66}$$

That is to say, in the thin-sheet limit the antisymmetric waves exhibit *no* dispersion! This surprising fact has been noted by Taylor for the case he studied, in which a/h was also small, and also by Crapper *et al.*

A rough explanation is that in the thin-sheet limit the fluid behaves like an elastic string of tension $2T$ and mass density $2\rho h$. However, for large amplitudes the analogy is apparently not precise. Our T is a fixed constant, whereas for a stretched string the tension would be a constant plus a term proportional to the local extension in length.

The shape of the surface in this limit is given by

$$\left.\begin{aligned} x &= (a/4k)\,[2E(\phi)-\phi],\\ y &= -\tfrac{1}{2}a\,\mathrm{cn}\,\phi. \end{aligned}\right\} \tag{67}$$

For $k^2 \geqslant \frac{1}{2}$, $a/\lambda \geqslant 0{\cdot}83$, the surface becomes vertical at certain points. At values $k^2 \geqslant 0{\cdot}73$, $a/\lambda \geqslant 2{\cdot}7$, the surface intersects itself, and the solution no longer admits a physical interpretation.

The limit $k\to 1$

It is interesting also to consider a third limit: $k\to 1$, corresponding to h/λ, $h/a\to 0$ with a/λ finite. We have just seen that this will be possible only for case I. For the dispersion formula we have

$$\kappa = 2,\quad \frac{a}{\lambda} = \tfrac{1}{2}\tan B,\quad c^2 = \frac{16T}{\rho}\left(\frac{a}{\lambda^2+4a^2}\right), \tag{68}$$

which shows a remarkably strong dependence of c on the amplitude. These are the 'essentially nonlinear' waves referred to in the introduction.

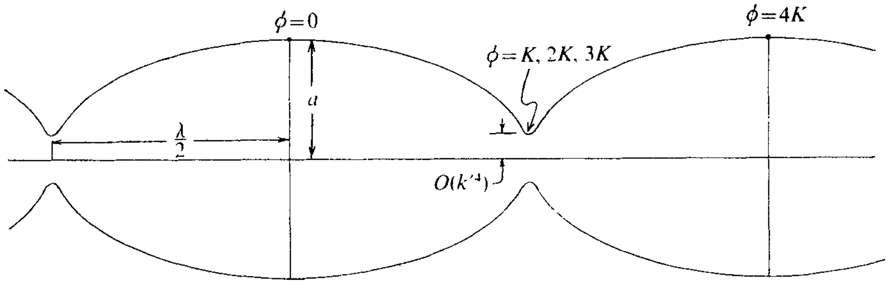

FIGURE 4. Case I profiles for $k = 1$.

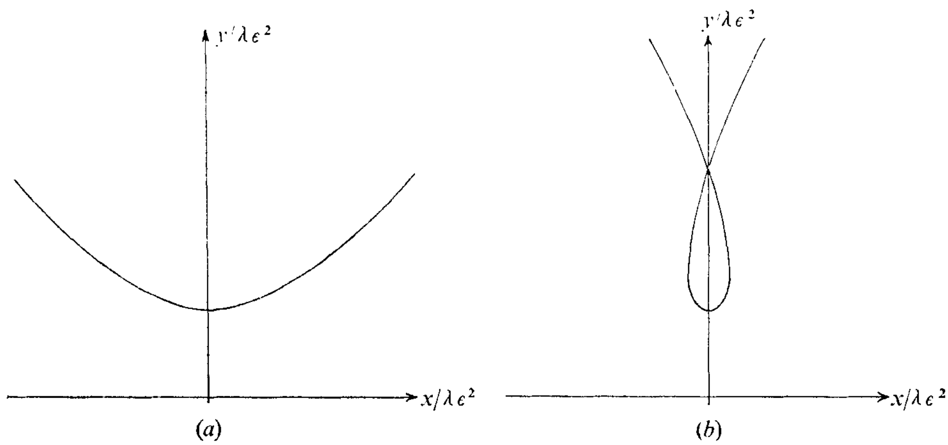

FIGURE 5. Profiles within the neck region. (a) $a/\lambda < \frac{1}{2}$. (b) $a/\lambda > \frac{1}{2}$.

From (33) and (34) we obtain the shape of the surface. This limit is somewhat more delicate to handle than the previous ones, because $K \to \infty$. For example, $\operatorname{sn}\phi$ becomes $\tanh\phi$ in the limit, but as $-\infty < \phi < +\infty$ only half of the wave is actually covered. To cover the entire wave we need to use at least two regions and match them together asymptotically. The straightforward limit $k \to 1$ gives

$$x = \frac{\lambda}{2}\left[\frac{\sinh\phi\cosh\phi}{\cosh^2\phi - \sin^2 B}\right], \quad y = a\left[\frac{\cos^2 B}{\cosh^2\phi - \sin^2 B}\right]. \tag{69}$$

Eliminating ϕ yields

$$\left(\frac{2x}{\lambda}\right)^2 = \left(1 - \frac{y}{a}\right)\left(1 + \frac{y}{a}\tan^2 B\right). \tag{70}$$

This is the equation of an ellipse. The surface will be a series of ellipses, as shown in figure 4. The sheet thickness at each neck is small but non-zero, of order k'^4. When $a/\lambda = \frac{1}{2}$ and $B = 45°$, the ellipses become semicircles and arrive at each neck with vertical slope. For $a/\lambda > \frac{1}{2}$ consecutive ellipses overlap, and the solution is once again unphysical. However even for $a/\lambda < \frac{1}{2}$ it is not clear that the solutions are everywhere physical, until we have examined the neck region in detail. As shown in figure 4, the curve spends an entire half-cycle near that point, and there could conceivably be a self-intersection or other singularity there.

Translate the exact solution by one half-cycle: $\phi = \phi_1 + 2K$. This gives

$$\left.\begin{aligned} x &= \frac{\lambda}{4\kappa}\left[2\kappa + 2E(\phi_1) - k'^2\phi_1 - 2k^2 \operatorname{sn}\phi_1 \operatorname{cd}\phi_1 - \frac{2kk'^2 \operatorname{sd}\phi_1 \operatorname{nd}\phi_1}{\operatorname{dn}B + k\operatorname{cd}\phi_1}\right], \\ y &= \frac{\lambda}{4\kappa}\left[(1+k^2)B - 2E(B) + \frac{2k'^2 \operatorname{sn}B \operatorname{cn}B}{\operatorname{dn}B + k\operatorname{cd}\phi_1}\right] \end{aligned}\right\} \tag{71}$$

and places the point of difficulty at $\phi = 0$. Letting $\epsilon = \frac{1}{8}k'^2$, we obtain in the limit $\epsilon \to 0$

$$\left.\begin{aligned} x &= \tfrac{1}{2}\lambda + \lambda\epsilon^2[\phi_1 + (1 - 2\sin^2 B)\sinh\phi_1 \cosh\phi_1], \\ y &= \lambda\epsilon^2[B + \sin B \cos B(2\cosh^2\phi_1 - 1)]. \end{aligned}\right\} \tag{72}$$

This is a smooth curve, as shown in figure 5. For $\phi_1 \to -\infty$, it matches the curve of (69) for $\phi \to +\infty$.

We have thus confirmed that, even in this extreme limit $k \to 1$, the entire surface for case I does remain non-intersecting and physically acceptable provided only that $a/\lambda < \frac{1}{2}$.

REFERENCES

BYRD, P. F. & FRIEDMAN, M. D. 1971 *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer.

CRAPPER, D. G. 1957 *J. Fluid Mech.* **2**, 532.

CRAPPER, D. G., DOMBROWSKI, N. & PYOTT, G. A. D. 1975 *Proc. Roy. Soc.* A **342**, 209.

TAYLOR, G. I. 1959 *Proc. Roy. Soc.* A **253**, 296.